Serie n°3 – September 24th

**Additional Exercises**

**Complex Numbers**
**Reciprocal spaces, X-ray diffraction**

### Exercise A1: irrationality of $\sqrt{2}$ and $\sqrt{n}$

A1a. We want to show that $\sqrt{2}$ is irrational, or in other words, there is no $(n,m) \in \mathbb{N}^2$, with $gcd(n,m) = 1$, such that $2 = \left(\frac{m}{n}\right)^2$.

(i) Show that 2 divides m.
(ii) Show then that 2 divides n.
(iii) Express a contradiction with $gcd(n,m) = 1$.

A1b. We want to show that $\forall n \in \mathbb{N}, \left(\sqrt{n} \in \mathbb{R} \setminus \mathbb{Q} \; or \; \sqrt{n} \in \mathbb{N}\right)$. We consider $n \in \mathbb{N}$ and suppose that $\sqrt{n} \in \mathbb{Q}$. Hence, $\exists (a,b) \in \mathbb{N}^2, \sqrt{n} = \frac{a}{b}$ and $gcd(a,b) = 1$. Consider a prime number $p$ in the factorization of $b$:

(i) Show that $p|a^2$, and hence necessarily we must have $p|a$.
(ii) Show that $b = 1$ and conclude.

### Exercise A2 : Inequalities of Cauchy-Schwartz and Minkowski

2a. For $n \in \mathbb{N}^*$ and $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}$, we define the function T of $\lambda \in \mathbb{R}$ such as

$$T(\lambda) = \sum_{i=1}^{n} (\lambda x_i + y_i)^2$$

(i) Show that $\forall \lambda \in \mathbb{R}, T(\lambda) \geq 0$.
(ii) Show that $T(\lambda) = \left(\sum_{i=1}^{n} x_i^2\right) \lambda^2 + 2\left(\sum_{i=1}^{n} x_i y_i\right)\lambda + \sum_{i=1}^{n} y_i^2$
(iii) Consider the equation $T(\lambda) = 0$ (assuming $\sum_{i=1}^{n} x_i^2 \neq 0$ ). What is the sign of the determinant ?
(iv) Deduce the inequality of Cauchy-Schwartz: $\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} {x_i}^2\right)\left(\sum_{i=1}^{n} {y_i}^2\right)$

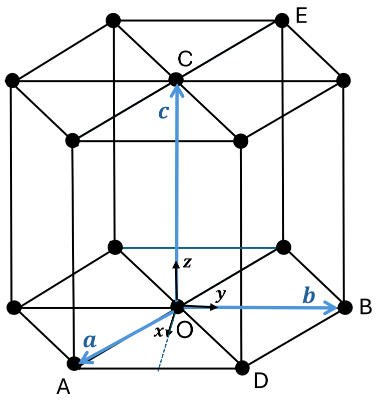
2b.

(i) Using 2a, show : $\sum_{i=1}^{n} (x_i + y_i)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right) + 2\sqrt{\sum_{i=1}^{n} {x_i}^2}\sqrt{\sum_{i=1}^{n} {y_i}^2} + \sum_{i=1}^{n} y_i^2$
(ii) Deduct the inequality of Minkowski: $\sqrt{\sum_{i=1}^{n} (x_i + y_i)^2} \leq \sqrt{\sum_{i=1}^{n} {x_i}^2} + \sqrt{\sum_{i=1}^{n} {y_i}^2}$

**Exercise A3: Distance between crystal planes in the hexagonal structure**

We consider the hexagonal structure shown to the right. We represented the origin, the orthonormal basis $\mathcal{B}_{(O,\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})}$ , and the Bravais lattice $\mathcal{B}_{(O,\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})}$, with:

- $\|\boldsymbol{a}\| = \|\boldsymbol{b}\| = a$, $\|\boldsymbol{c}\| = c$, and $a \neq c$ (where $\|\boldsymbol{a}\|$ is the norm of the vector $\boldsymbol{a}$);
- $\widehat{(\boldsymbol{a},\boldsymbol{b})} = \frac{2\pi}{3}$, $\widehat{(\boldsymbol{a},\boldsymbol{c})} = \widehat{(\boldsymbol{b},\boldsymbol{c})} = \frac{\pi}{2}$ (where $\widehat{(\boldsymbol{a},\boldsymbol{b})}$ is the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$).




A3a.

(i) What are the coordinates of the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ in the basis $\mathcal{B}_{(O,\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})}$ ?

(ii) What are the coordinates in the basis $\mathcal{B}_{(O,\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})}$ and $\mathcal{B}_{(O,\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})}$ of the points A,B, C, D and E shown on the schematic ?

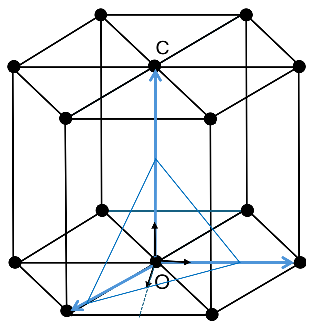
A3b.

(i) What is the volume of the cell defined by the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$?

(ii) Show that the reciprocal lattice vectors are given by, in the $\mathcal{B}_{(O,\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})}$ basis:

$$\boldsymbol{a}^* = \frac{4\pi}{a\sqrt{3}}\boldsymbol{x};\ \boldsymbol{b}^* = \frac{4\pi}{a\sqrt{3}}\left(\frac{1}{2}\boldsymbol{x} + \frac{\sqrt{3}}{2}\boldsymbol{y}\right);\ \boldsymbol{c}^* = \frac{2\pi}{c}\boldsymbol{z}$$

A3c. We consider the crystal plane (hkl) (h,k,l three relative integers) shown on the schematic, that intercepts the $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ axis at positions $\frac{\boldsymbol{a}}{h}, \frac{\boldsymbol{b}}{k}, \frac{\boldsymbol{c}}{l}$ respectively.




(i) What is the normal to the plane in the orthonormal $\mathcal{B}_{(O,\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})}$ basis ?

(ii) Show that the equation of the plane in $\mathcal{B}_{(O,\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})}$ is given by:

$$\mathcal{P} = \left\{ M\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \frac{(2h+k)}{a\sqrt{3}}x + \frac{k}{a}y + \frac{l}{c}z = 1 \right\}$$

4d. Assuming that the closest (hkl) plane parallel to $\mathcal{P}$ is the one passing through the origin, show that the distance between the $(hkl)$ plane is given by:

$$d_{(hkl)} = \frac{1}{\sqrt{\frac{4}{3a^2}(h^2 + k^2 + hk) + \frac{l^2}{c^2}}}$$